Conway's Rational Tangles

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1 Introduction

I learned about this "trick" in a lecture by John Conway a number of years ago. He calls it "Rational Tangles" and there is plenty of information about it on the internet. Since then I have used it myself in classrooms of students in middle schools and older. The underlying mathematics is very interesting, but it is not necessary that the students understand the mathematics to get a lot out of the trick.

This document is intended for teachers and includes some pedagogical advice on how to use Conway's trick to teach the students something about mathematics.

The idea is that we can associate a number with a tangle of two ropes and that by performing a sequence of two simple operations, we can untangle the ropes in a straightforward way.

Often the best way to get students to practice rote mathematics is to give them a problem that is intrinsically very interesting, but whose solution requires repeated calculations of the sort that you are trying to get them to practice. To do these rope trick calculations, the students will need to practice arithmetic with fractions and to work with positive and negative numbers.

2 Getting Started

To demonstrate the trick, you need four students and two lengths of rope that are about 10 feet long. Heavier rope is better because it is easier to see the knot structure and it is harder to accidentally pull into tight knots that are difficult to work with. Bring at least one, but better more than one, plastic shopping bags.

Get four volunteers to stand at the corners of a rectangle at the front of the class with each student holding one end of a rope. In the initial configuration, the two ropes are parallel to each other and parallel to the front row of seats in the classroom. In Figure 1, the top pair of parallel lines represents the two ropes, and the small circles at the ends with the letters "A". "B", "C" and "D" represent the students. If you imagine that you are looking down on the students from the ceiling, the rest of the class is seated above the entire figure.

It's a good idea to make sure that each student has a solid grip on the rope, perhaps wrapping the end once around hand so that it is not accidentally dropped. During the trick, no student should ever let go of his or her end of the rope. Don't let the kids start jerking on the rope, since if one end comes loose, it is very easy to lose track of exactly how the ropes were tangled, and if this occurs, the trick will fail, and will become completely non-interesting.

You can explain to the kids that they are going to do something like a square dance where the four students perform one of three "figures". Also explain that the initial configuration with the parallel, untangled ropes will be assigned the number zero, and that each figure will affect that number in a fixed way. Also, tell the rest of the kids in the class to pay attention, since you'll swap out sets of kids from time to time so that many more of them can be part of the action.

The only thing that matters is the configuration of the ropes: which student is in which position does not affect the tangle's number.

3 The Basic Figures

Conway calls the two main figures "Twist 'em up" and "Turn 'em around". The unfortunate thing about this choice is that they both begin with the letter "T". If you're trying to analyze the results of various sequences, there's no easy shorthand. Perhaps "Twist" and "Rotate" are better, since then you can write something like "TTRTR" to indicate that sequence of 5 figures in the dance. In what follows, I will use the names **Twist** and **Rotate**, and "T" and "R" as shorthand, especially when there is a sequence of moves.

When explaining the **Twist**, make sure that all four students pay attention, since although only two of them perform any particular **Twist**, they may be arranged differently later in the dance and will have to do it when they are in those positions.

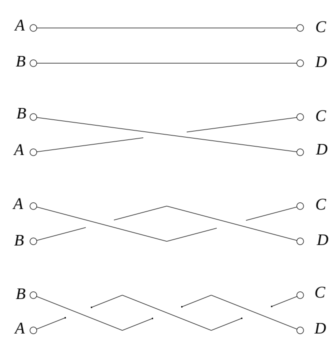

Figure 1: Twisting

To perform the **Twist** figure, the two students on the left change places, with the student initially in the rear lifting his or her rope and the student in front stepping under it. As it is labeled in the figure, students A and B swap places, where B lifts his or her rope and A steps under it to the rear. In Figure 1 the results of performing zero, one, two and three of the **Twist** figures are shown from top to bottom. Notice that from the points of view of the students holding the ropes, after starting from zero and performing just Twist moves, the ropes will appear to twist away from them in a clockwise direction, and this is true of all the rope-holders. Notice that with each **Twist**, the positions of students A and B are swapped. Demonstrate this to the class.

Each time a **Twist** move is made, the number associated with the tangle is increased by 1, so in Figure 1 the four tangles from top to bottom are represented by the numbers 0, 1, 2 and 3. If the tangle's number is a pure integer like this, then the integer represents the number of half-twists in the rope.

We will not allow an **UnTwist** figure in this trick, but if there were one, it would be easy to do: the same two people on the left change places, but this time the person in front raises the rope and the person behind steps under it. Such an **UnTwist** figure would subtract 1 from the tangle's number. This is a very obvious concept, and if **Twist** and **UnTwist** were the only two legal moves, it's clear that starting from zero, any positive or negative integer could be obtained, and if you knew that number, the ropes could be untangled by performing that number of **UnTwist**s or **Twist**s, depending on whether the number were positive or negative.

The second figure **Display**, does nothing to the tangle; it is simply to display the condition of the ropes to the rest of the class. To do a **Display**, the two people farthest from the class raise their ropes and the two in front lower them so the tangle is displayed in an unobstructed way.

The third figure, **Rotate**, has each student move one position clockwise, when viewed from above. In Figure 1 if we began from the top arrangment in the figure, a **Rotate** would move A to C's position, B to A's position, D to B's position and C to D's position. If you were to **Rotate** four times in a row, each student would wind up where they started. Demonstrate to the class that at least when there are only twists in the rope (and in fact it will always be true) that two **Rotate**s will return the ropes back to where they started, even though the students will be on the opposite sides. Perhaps this can be made clear by reminding the students that the number 3, for example, represents 3 clockwise half-twists of the ropes from the point of view of any of the students. As they turn around, nothing is going to change the clockwise orientation, so after the two pairs on the ends have swapped places, they still see three clockwise half-twists, so rope configuration is unchanged.

This observation indicates that the operation on the number associated with the tangle has to bring it back to where it started if you apply that operation twice.

For another clue about what this might be, have the students do this: Start from the ropes in a "zero" tangle. Do one **Twist** (so the number is now 1). Next do a **Rotate**. Finally do another **Twist**, and the ropes are untangled. This means that after the **Rotate**, the number must have been -1, since adding 1 to it brings us back to a 0 configuration. So **Rotate** changes a 1 to a -1.

The class will then probably make the reasonable (but wrong) guess that a **Rotate** move multiples the number by -1. This makes sense, since multiplying by -1 twice returns to the initial number. But it's easy to test: Start from 0, do two **Twist**s followed by a **Rotate**. If **Rotate** multiplies by -1, then the ropes should be in the -2 state, and two **Twist**s should undo that. Try this, and see that they do not.

Depending on the sophistication of the class, you can either tell them the answer, or try to lead them to it by considering other operations that turn 1 into -1 but not 2 into -2, yet when repeated twice bring every number back to itself.

The correct answer is that **Rotate** takes the tangle represented by x and turns it into the tangle represented by $-1/x$. Thus, starting from zero, the sequence TTR leaves a tangle with value $-1/2$. You can check this by starting from zero, doing a TTR (which should leave $-1/2$), then doing a T (yielding a value of $-1/2+1=+1/2$), then an R (yielding -2) and then a TT brings you back to zero.

See if the kids can figure out how to get a TTT (with value 3) back to zero. It's a little bit complicated and requires 8 steps:

$$3 \xrightarrow{R} -\frac{1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{R} -\frac{3}{2} \xrightarrow{TT} \frac{1}{2} \xrightarrow{R} -2 \xrightarrow{TT} 0.$$

4 Getting to Zero

At this point you can begin to consider how an arbitrary twisting can be converted back to the untangled state using only the **Twist** and **Rotate** commands. You might begin by having the kids suggest moves that tangle the rope a little bit: perhaps seven or eight steps, but carefully keeping track of the numbers. For now, make sure they start with at least two **Twist**s and mix in both **Twist**s and **Rotate**s after that.

If they try to do two **Rotate**s in a row, point out that although this is perfectly legal, the second **Rotate** undoes the first, so the two moves taken together achieve nothing.

What you can do is collect move suggestions from different kids, keeping track of the tangle's number until it is "suitably" complex. Also, as a first example, stop them after a **Twist** command has left the number positive. For example, the sequence TTRTTTRT, starting from zero, leaves a tangle with associated number $3/5$. This is a nice number since it doesn't have numerator or denominator that's too big, but it is complicated enough to be interesting. We'll use this example in what follows.

Tell the kids that their goal is to get the number $3/5$ down to zero using only **Rotate** and **Twist** commands. As a first hint, tell them that **Twist** will add 1 which will take the number even farther away from zero, so to make progress, the only possible command is **Rotate**. We now have $-5/3$.

Point out now that another **Rotate** will just undo the one they did, so the only reasonable next step is a **Twist**, yielding $-2/3$. At this point a **Rotate** will not put you back where you started, but it would yield a positive number, and that can't be good, since another **Rotate** is useless, and one or more **Twist** commands would take the number away from zero. Thus from $-2/3$, the only reasonable move is another **Twist**, yielding $1/3$.

Repeating the arguments above, we clearly need a **Rotate**, taking us to -3, and then three **Twist**s get back to zero. Go ahead and do this with the ropes and verify that indeed it does untangle the mess.

If you haven't done it already, this is a good time to swap in a new set of four students.

Now for the best part: make another tangle, a bit more involved than the last one, andonce it's created, put the tangle into a bag as follows. Take a plastic bag and make two small holes in the corners opposite the opening. Take the ropes, one at a time, from the two kids on the left and feed them through the holes and back to the kid. Pull the bag opening over the tangle and tie the whole thing shut so that the tangle is completely enclosed in the bag.

Finally, carefully apply the steps that undo the tangle and when you're done, there will be a horrible snarl of ropes and plastic, which, if you've made no mistakes, should be equivalent to zero. To prove it, tear the plastic bag into pieces to extract it from the tangle, and then with a few tugs, the entire mess will appear to magically untangle itself!

It's sometimes fun to do this more than once, so bring more than one plastic bag to the class.

5 Discussion Topics

Here are a few ideas that may lead to interesting class discussions:

5.1 Infinity as a Tangle Number

Try starting with zero and do a single **Rotate**. This yields the nonsense value $-1/0$, but it's not a nonsense tangle. Another **Rotate** will bring it back to zero, and in fact, it sort of behaves like "infinity" in the sense that a **Twist** (try it) leaves it exactly the same. This sort of corresponds to the idea that adding 1 to ∞ leaves it unchanged.

5.2 Proof of Convergence to Zero

Can you prove that the scheme outlined above will always eventually grind any initial fraction down to zero? Go through a few examples and see what is happening. Here is an example starting from $-5/17$:

$$-\frac{5}{17} \xrightarrow{T} \frac{12}{17} \xrightarrow{R} -\frac{17}{12} \xrightarrow{TT} \frac{7}{12} \xrightarrow{R} -\frac{12}{7} \xrightarrow{TT} \frac{2}{7} \xrightarrow{R} -\frac{7}{2} \xrightarrow{TTTT} \frac{1}{2} \xrightarrow{R} -\frac{2}{1} \xrightarrow{TT} 0.$$

Note that after each **Rotate** command, the resulting negative fraction has a smaller denominator. Why is this? If the denominators always eventually get smaller, they must eventually get to 1. But that will be a negative integer, and we know that that number of **Twist** commands will reduce it to zero.

5.3 Relationship to the Greatest Common Denominator

If the students are a bit advanced, you can point out that the process of reducing the fraction down to zero is almost exactly the same as finding the greatest common divisor (the GCD) of the numerator and denominator. Since we begin with a fraction reduced to lowest terms, this will always get us down to 1 as the GCD.

Euclid's algorithm for calculating the GCD of two numbers works as follows. If the two numbers are m and n, and supposing that $m > n$, we can write: $m = kn + l$, where k is an integer and $|l| < n$. Any number that divides m and n must divide l in the equation above, so we can conclude that $GCD(m, n) = GCD(n, l)$. The numbers in the right hand side are reduced, and the process can be repeated until one is a multiple of the other.

Here is an example: find the GCD of 4004 and 700:

$$\begin{aligned}
4004 &= 700 \times 5 + 504 \\
700 &= 504 \times 1 + 196 \\
504 &= 196 \times 2 + 112 \\
196 &= 112 \times 1 + 84 \\
112 &= 84 \times 1 + 28 \\
84 &= 28 \times 3.
\end{aligned}$$

The GCD of 4004 and 700 must divide 504 from the first line, so $GCD(4004, 700) = GCD(700, 504)$. The same process can be continued to obtain:

$$\begin{aligned}
&GCD(4004, 700) = GCD(700, 504) = GCD(504, 196) \\
&\quad = GCD(196, 112) = GCD(112, 84) = GCD(84, 28).
\end{aligned}$$

But 84 is an exact multiple of 28, so $GCD(84, 28) = 28$, and we can therefore conclude that $GCD(4004, 700) = 28$ as well.

Note that there is no requirement that the numbers on the right hand sides of the sequence of reductions be positive. All that we require for convergence is that they be smaller in absolute value than the smaller of the two values for which you are trying to obtain the GCD. Also note that division can be achieved by repeated subtraction, and if you simply check to see if the subtraction yields a non-positive number, you know you have gone far enough. It's sort of like backing up your car until you hear breaking glass, but it works!

With all that in mind, let's find the GCD of 5 and 17 using this totally crude method:

$$\begin{aligned}
5 &= 17 \times 1 - 12 \\
17 &= 12 \times 1 + 5 = 12 \times 2 - 7 \\
12 &= 7 \times 1 + 5 = 7 \times 2 - 2 \\
7 &= 2 \times 1 + 5 = 2 \times 2 + 3 = 2 \times 3 + 1 = 2 \times 4 - 1 \\
2 &= 1 \times 1 + 1 = 1 \times 2 + 0.
\end{aligned}$$

Note the similarity of this method to the one we used to obtain $GCD(4004, 700)$ above. But this time, rather than doing a division, we do repeated subtractions until the remainder is zero or negative. Then we use the (positive value of) the remainder in the next step. We finally discover that 1 divides 2 evenly, so $GCD(5, 17) = 1$. Now compare this sequence to the one that reduces the tangle value $-5/17$ to zero at the beginning of this section. You will see that the calculations are virtually identical.

5.4 What Tangle Numbers Are Possible?

Is it possible to start from zero and get to any (positive or negative) fraction? Have the students mess around for a while and see what fractions they can come up with. Also, set goals, such as, "Can you start from zero and get to -3?" If there is no progress, here is a giant hint:

$$\frac{3}{1} \xrightarrow{R} -\frac{1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{R} -\frac{3}{2} \xrightarrow{TT} \frac{1}{2} \xrightarrow{R} -\frac{2}{1} \xrightarrow{TT} 0$$

$$0 \xrightarrow{TT} \frac{2}{1} \xrightarrow{R} -\frac{1}{2} \xrightarrow{TT} \frac{3}{2} \xrightarrow{R} -\frac{2}{3} \xrightarrow{T} \frac{1}{3} \xrightarrow{R} -\frac{3}{1}.$$

If we start from 3 and work our way to zero using the methods we have used before, the sequence RTRTTRTT does the trick. But now note that if we start from zero and use the reverse of the sequence above, namely: TTRTTRTR, we get to -3. Also, note that at every stage in the sequence, the same fractions are generated, except that they have opposite signs.