

The Puppies and Kittens Game is courtesy of The Teacher's Circle: Finite Games by Paul Zeitz. The alternative comes courtesy of AMC from Steve Dunbar.

The Puppies and Kittens Game

For this game, two players alternate turns. We start with a pile of 7 kittens and 10 puppies. A legal move is one of the following three possibilities: removing any positive number of puppies (but no kittens), or any positive number of kittens (but no puppies), or an equal number of both puppies and kittens. The winner is the last player who makes a legal move. Which player has a winning strategy? For what starting values of k kittens and p puppies will player 1 win? For what values of k kittens and p puppies will player 2 win?

Alternative Statement

A queen is placed on a lattice point in the first quadrant. In turns, two players move the queen according to the following rule: if the queen is at a lattice point (x, y) , it can be moved to any first quadrant lattice point $(x - n, y)$, $(x, y - n)$, or $(x - n, y - n)$ with $n \in \mathbb{N}$. If a player cannot move, he loses. For what initial positions of the queen does the second player win?

Solution:

We will answer the last questions first and then describe the winning strategy for the case of 7 kittens and 10 puppies. We start with small numbers of puppies and kittens and look for a pattern. Let p be the number of puppies and k be the number of kittens. If $p = 0$, $k = 0$, or $p = k$, then by the rules of the game, player 1 wins.

What are the smallest values for p and k so that player 2 wins? If you have 2 kittens and 1 puppy, player 2 will win no matter what player 1 does. If player 1 takes the puppy, player 2 will win by taking the kittens and likewise player 2 wins if player 1 takes all the kittens. If player 1 takes 1 puppy and 1 kitten, player 2 wins by taking the remaining kitten. Similarly, if you start with 2 puppies and 1 kitten, player 2 will win.

Now something interesting happens. What if you start with 3 puppies and 1 kitten? Then you are only one move away from having 2 puppies and 1 kitten, which is a case we have examined. So player 1 has a winning strategy. If you start with $p = 3$, $k = 1$, player 1 wants to take 1 puppy away. This forces player 2 to be the first player to play the $p = 2$, $k = 1$ round and we know he/she is tragically doomed. Following this train of thought, player 1 will have a winning strategy for $k = 1$ and any number of puppies $p \geq 2$. Similarly, player 1 will win for $p = 2$ and any number of kittens $k \geq 1$. There's also the rule about taking an equal number of puppies and kittens in a turn. So player 1 will win if you start with $k = 2$, $p = 3$; $k = 3$, $p = 4$; $k = 4$, $p = 5$; . . . in general $(k, k + 1)$ for $k \geq 2$. Following this logic, you will find that $k = 4$, $p = 7$ is the next time that player 2 has the winning strategy.

In general, you can graph out the various scenarios:

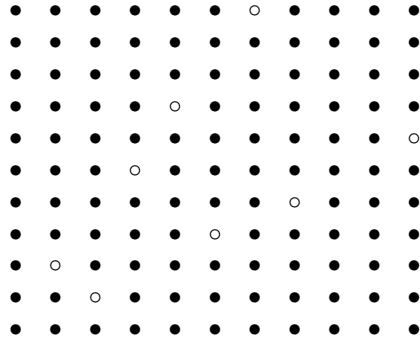


FIGURE 1. The Winning Scenarios for Player 1 and Player 2

Here, ● = space where player 1 has the winning strategy, ○ = space where player 2 has the winning strategy.

So we return to our original question: if we start with 7 kittens and 10 puppies, who has a winning strategy? Looking at our graph, $(7, 10)$ is a black dot, so player 1 has a winning strategy. On each turn, his/her winning strategy is to take the appropriate number of puppies and kittens so that he/she lands on a white circle. So basically, player 1 wants to play “leap frog” to the white circles.

Inevitably, one must ask oneself, how do you predict where the white dots are? This is where the math “picks up” a bit. We can establish the following pattern (here we are looking at the location for half of the white dots — we can switch the kittens and puppies to get the other half of the white dots):

Difference	Kittens	Puppies
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
\vdots	\vdots	\vdots
n	$\lfloor \phi n \rfloor$	$\lfloor \phi n \rfloor + n$

TABLE 1. Locations of white dots

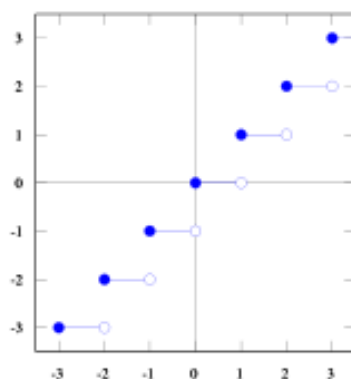


FIGURE 2. The graph of the Floor Function

Here, $\phi = \frac{1 + \sqrt{5}}{2}$ is the Golden Ratio (you might remember the Golden Ratio from studying the Fibonacci Number or looking at Golden Rectangles) and $\lfloor x \rfloor$ is the floor function of x . Formally, that means

$$\lfloor x \rfloor = \max \{n \in \mathbb{Z} : n \leq x\}.$$

So $\lfloor 2.7 \rfloor = 2$, $\lfloor 13.5 \rfloor = 13$, and $\lfloor -1.4 \rfloor = -2$. A graph of the floor function is shown in Figure 2.

For a more full appreciation of the math that is going on here, we are using Beatty's Theorem:

THEOREM 0.1 (Beatty's Theorem). *If α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then $\{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \dots\}$ and $\{\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, \dots\}$ are disjoint and their union is all integers.*

For a version of the proof and other fun facts, see <http://www.cut-the-knot.org/proofs/Beatty.shtml>.