Decimal Representation of Rational Numbers

Question: What is the decimal representation of 1/17?

Obtaining the answer may be painful; but there is mathematical pain medication. First let's consider some easier questions, settle on the terminology, and see what tools are available.

1. Which of the following rational numbers have a non-terminating decimal representation?

$\frac{3}{5}, \frac{4}{11}, \frac{7}{20}, \frac{9}{6}, \frac{12}{6}, \frac{9}{7}, \frac{2}{17}$

How can you tell? Conjecture:

Definition: A period of a non-terminating decimal is the number of digits in the repetend.

2. What are the periods of $\frac{9}{7}$ $\frac{1}{11}$ $\frac{1}{13}$ $\frac{1}{17}$?

Calculators can answer this question for some fractions. What can we do if calculators aren't enough?

We need more understanding of this process, if we really understand the process we can generalize and make a conjecture to help us with the more complicated rational numbers.

Find the decimal representation of 9/7 using long division and demonstrate this using base 10 blocks.

Long Division	Division Algorithm n = q*d + r	Exact Fraction Expansion
7|9.000000000	9 =1*7 + 2	9/7 = 1 + 2/7

3. Is there a bound on the period of a rational number with divisor d? What determines the period? Call the answer to this Fact 1.

F1:

4. How do you use your calculator to compute the remainder? Use 100 ÷ 17 as an example. Call this answer Fact 2.

F2: To compute the remainder of n/d:

5. How do you use your calculator to compute the 8th remainder? Let's look at the process we used to get the remainders.

$$\frac{n}{d} = q_0 + \frac{r_0}{d}$$

$$\frac{10r_0}{d} = q_1 + \frac{r_1}{d}$$

$$\frac{10r_1}{d} = q_2 + \frac{r_2}{d}$$

etc …

Now we can cut out the middle man,

$$\frac{10n}{d} = 10q_0 + \frac{10r_1}{d} = 10q_0 + q_1 + \frac{r_1}{d}$$

$$\frac{10^2n}{d} = 10^2q_0 + 10q_1 + \frac{10r_1}{d} = 10^2q_0 + 10q_1 + q_2 + \frac{r_2}{d}$$

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$$\frac{10^l n}{d} = 10^l q_0 + 10^{l-1}q_1 + 10^{l-2}q_2 + \cdots + q_l + \frac{r_l}{d}$$

The above equations indicated that, for example, the 8th remainder r_8 is the same as the remainder you get from computing, $\frac{10^8n}{d}$; or we can say that $\frac{10r_7}{d}$ and $\frac{10^8n}{d}$ have the same remainders; or $\frac{10r_{k-1}}{d}$ and $\frac{10^kn}{d}$ have the same remainder, r_k, where k is any positive integer. Even more: to say, for example, that r_3 is the remainder of $\frac{10^3n}{d}$ means that $\frac{10^3n}{d} - \frac{r_3}{d} = integer$. So an integer must also = $10^5(\frac{10^3n}{d} - \frac{r_3}{d}) = \frac{10^8n}{d} - \frac{10^5r_3}{d}$. So $\frac{10^5r_3}{d}$ and $\frac{10^8n}{d}$ must have the same remainder, r_8. We sum this up.

F3: The following all have the same remainder r_k: $\frac{10r_{k-1}}{d}, \frac{10^kn}{d}, and\, \frac{10^{k-j}r_j}{d}$

5. What is the 7th remainder, r_7 for: 3/13, 1/17, 2/19.

6. The remainder r_{14} for 1/17 is 8, so the remainders r_{15} and r_{16} are ________ and the decimal digits 15 and 16 (q_{15}, q_{16}) are _____________ .

Are there "easy" ways to determine the period, p for n/d (assume n/d in lowest terms)?

The period of n/d is the number of distinct, successive remainders. That is, for 2/7, remainder r_8 and r_{14} must be equal. For n/d, as soon as we get an r_l that is equal to an earlier r_k then p = *l*-*k* is the period. This produces the condition we need to compute the period. This is because F3 tells us that $r_l = r_k$ is the same as saying that $\frac{10^l n}{d}$ and $\frac{10^k n}{d}$ have the same remainder; or, equivalently, that $\frac{10^l n}{d} - \frac{10^k n}{d} =$ an integer; or, equivalently, that for some integer M

$$(10^l - 10^k)n = Md$$

So

$$10^k n\left(10^{l-k} - 1\right) = Md$$

Or

$$10^k n(10^p - 1) = Md,$$

where p is the period. Now the prime factors for d must appear on the left hand side of the equation. None are part of the factors of n, since n/d is in lowest terms. So they must be part of the factors for 10^k and (10^p – 1). The only prime divisors of 10^k are 2 and 5. Neither of these divide 10^p – 1, because, for example, 10^p – 1 divided by 5 is an integer – 1/5. So d = st, where s is either 1 or some product of 2's and 5's and t is some integer > 1 that is not divisible by 2 or 5 (if t = 1, then n/d would be terminating, and we're assuming n/d is not terminating). So t must divide 10^p – 1. That is 10^p/t = integer + 1/t; and this means 10^p/t has remainder 1. We sum this up.

F4: Suppose n/d is in lowest terms and that d = st, where s is either 1 or is some product of 2's and 5's and the integer t > 1 and is not divisible by 2 or 5. Then the decimal representation for n/d is periodic with period p, where p is the 1st integer that has 1 as the remainder for 10^p/t.

7. Find all primes d so that n/d (lowest terms) has period 1? Period 2? Period 3? Period 6?

8. What is the period of 1/17? (Here it helps to use: the remainder for 10^k/d = the kth remainder, r_k, for 1/d = the remainder for $10r_{k-1}$/d, to keep numbers small enough for calculator computations.) (Show on spreadsheet. Even better is to get 8 digits at a time using 10^8 r_0/d has remainder r_8 and the integer part is the 1st 8 decimal digits, then you can continue getting the next 8 digits with 10^8 r_8.)

Extension:

What if the divisor is not prime?

What if it is a non-unit fraction?

Using F3 and F4 explain why 5/18 and 2/3 have the same period.

Some additional Facts:

1. If d is prime that the period must divide d – 1.

2. In general the period must divide the number of numbers < d that are relatively prime to d (count the number 1).

3. If 10^k mod(d) =r_k and 10^l mod(d) = r_l, then 10^{k+l} mod(d) = $r_k r_l$ mod(d).

For example, to determine the period of 1/71, (2) indicates we need only consider p that divide 70 = 2*5*7. That is, we need only consider p = 2, 5, 7, 10, 14, 35, 70. Using (3), we compute the remainders for 10/71, 10^2/71, 10^3/71 and get 10 mod(71) = 10, 10^2 mod(71) = 29.

Thus 10^3 mod(71) = 10*10^2 mod(71) = 10*29 mod(71) = 290 mod(71) = 6,
and 10^5 mod(71) = 10^2*10^3 mod(71) = 29*6 mod(71) = 32,
and 10^7 mod(71) = 10^{3+3+1} mod(71) = 6*6*10 mod(71) = 5,
and 10^{10} mod(71) = 10^{7+3} mod(71) = 5*6 mod(71) = 30,
and 10^{14} mod(71) = 10^{7+7} mod(71) = 5*5 mod(71) = 25,
and 10^{35} mod(71) = $10^{14+14+7}$ mod(71) = 25*25*5 mod(71) = 1.

So 1/71 has period p = 35. Try this for 1/97.

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