The Hat Problem

Stephen Hartke & Julie Kreizel

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The Setup: One team of three contestants are in a room.

A **red** or **blue** hat is randomly put on each contestant;
each contestant can see the hats of everyone *else*
but *not* his/her own.

The Game: Each contestant must (*simultaneously*)

1. Guess the color of *his/her* hat,
2. Or *pass*.

To Win: *At least one* contestant guessed correctly,
and *no one* guessed incorrectly.

The team can confer on a strategy beforehand.

What strategy is best?

Strategies

Pick a team **captain**.

The captain guesses red/blue randomly.

everyone else *passes*.

This strategy wins 50% of the time.

Observation

For each individual contestant,

his/her *guesses* will be wrong **half** the time.

Can we group the wrong guesses together?

Another Strategy

Each contestant does:

- If the other two hats are different colors, pass.
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How often does this strategy win?

Analysis

Hats	Guesses	Win?
000	111	no
100	1xx	yes
010	x1x	yes
001	xx1	yes
110	xx0	yes
101	x0x	yes
011	0xx	yes
111	000	no

This strategy wins $\frac{6}{8} = 75\%$ of the time!

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Could we do better?

Best Possible

There is no strategy that wins more than 75% of the time.

To win $\frac{7}{8}$ of the time, need to have 7 right guesses.

But then have 7 wrong guesses!

But only one outcome loses, which has ≤ 3 wrong guesses.

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The **hypercube** Q_n has as vertices
all of the binary strings of length n.

Two strings are connected by an edge
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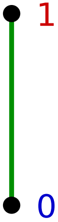

Q_1

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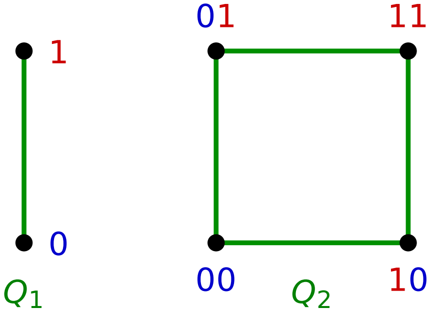

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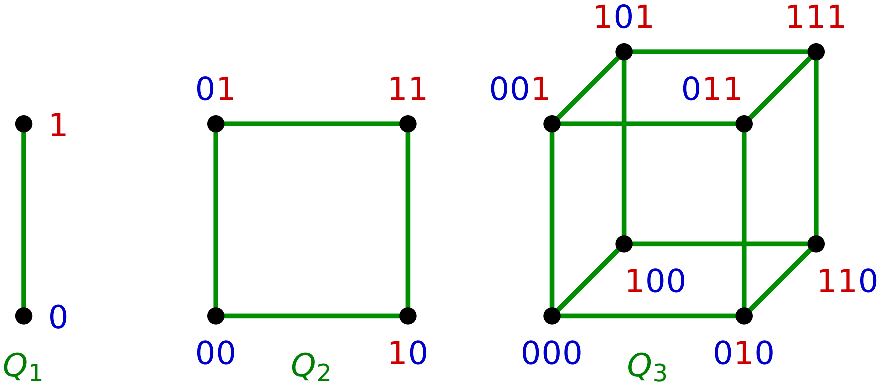

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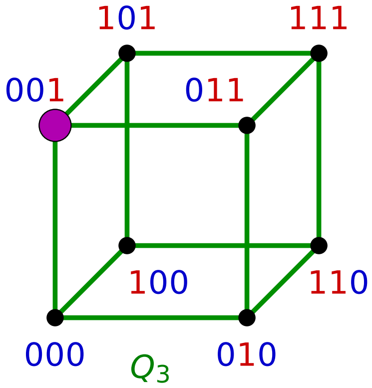
The colors of the placed hats give a vertex in the hypercube.



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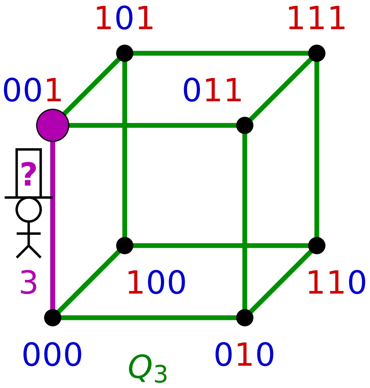
But each contestant only sees the *other* hats—

This determines an edge for each.



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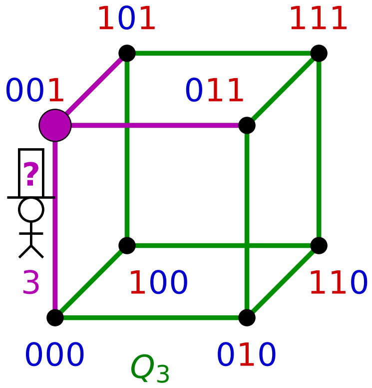
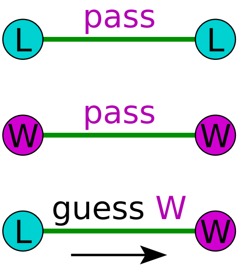
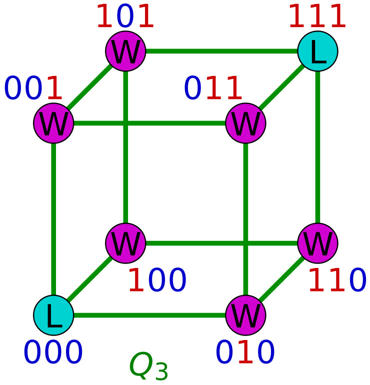


Geometric Interpretation

Label the vertices **W** or **L**.

Have each contestant:



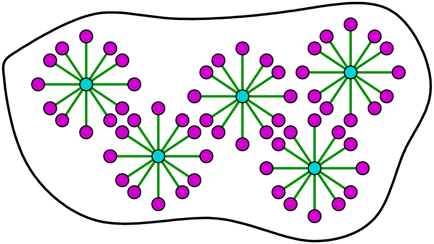


Label the Hypercube

We want a labeling of Q_n with **W** and **L** so that:

1. No two **L**s are next to each other.

2. Each **W** is next to exactly one **L**.



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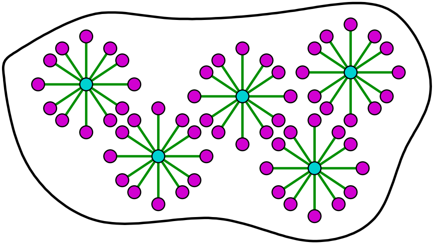

Can be done using error-correcting codes when $n = 2^k - 1$.

Gives probability of winning of $\dfrac{n}{n+1}$.

Hamming Codes

We operate modulo 2, so that $1 + 1 = 0$.

The Hamming code is the set of solutions

to a system of linear equations (modulo 2):

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The only solutions are 111 and 000 — these were the **L**s!

For $n = 7$ Players

To find the **L**s, we solve the system

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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There are 16 solutions:

0000000	0001110	0010111	0011001
0100011	0101101	0110100	0111010
1000101	1001011	1010010	1011100
1100110	1101000	1110001	1111111

When $n \neq 2^k - 1$

The exact answer is not known when $n \neq 2^k - 1$,

and this is an active area of research!

Todd Ebert introduced The Hat Problem in his 1998 thesis.

More sources are available on the web!