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M. D. Atkinson; W. F. Lunnon

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one: it is easily seen that the property stated above holds also for rotational symmetry; therefore, since O_{ij} is the rotation centre of the circle C_{ij}, P_{uv} is a rotation plane of the torus. So we have:

A torus is a figure generated in two different rotations.

This fact, certainly rather strange, emphasises the complete symmetry of the torus in $\mathbb{R}^4$. Moreover, it can be shown that there is no other rotation plane.

For the next remark it is convenient to consider a torus inscribed in the hypercube $\{\mathbf{x} | -1 \leqslant x_i \leqslant +1$ for every $i\}$, for instance the one with equations $x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1$. By examining the equations, one can immediately deduce that the torus is entirely contained in the hypersphere with equation $\sum_i x_i^2 = 2$. In fact, this hypersphere can be proved to be divided by the torus into two congruent 3-dimensional manifolds, and each of these is homeomorphic to the set of the interior points of a torus in $\mathbb{R}^3$.

This, too, confirms that in $\mathbb{R}^4$ there is a model of the torus which is more regular, and nicer, than any model in $\mathbb{R}^3$.

CLAUDIO BERNARDI
MANUELA MOSCUCCI

Università di Siena, Istituto di Matematica, via del Capitano 15, 53100 Siena, Italy

Regular fault-free rectangles

M. D. ATKINSON AND W. F. LUNNON

Provided one of the integers m and n is even, it is obvious that an $m \times n$ rectangle can be covered with $\frac{1}{2}mn$ dominoes, each domino covering two unit squares. If some of the dominoes are oriented in each of the two possible ways, it is possible that there may be no continuous straight line of domino edges, either horizontal or vertical, across the rectangle: in other words, each of the $(m - 1) + (n - 1)$ fault lines on the underlying grid is crossed by at least one domino. Such rectangles are called *fault-free*, and are mentioned by S. Golomb in [1] where the concept is attributed to R. I. Jewett. An example with $m = 6$, $n = 8$ is shown in Fig. 1.

This example has the further property that each of the horizontal fault lines is crossed by the same number of dominoes and each of the vertical fault lines is crossed by the same number (2 in each case). In general we shall say that a rectangle with vertical and horizontal sides of integral lengths m and n respectively and with a covering of $\frac{1}{2}mn$ dominoes is a *regular fault-free rectangle* with parameters m, n, k, l if there exist positive integers k, l such that each horizontal fault line is crossed by precisely k

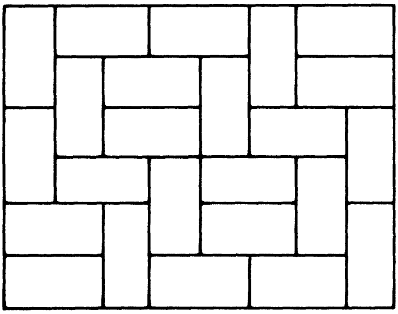

FIGURE 1.

dominoes and each vertical fault line is crossed by precisely l dominoes. Fig. 2 shows a further example with $(m, n, k, l) = (10, 14, 2, 4)$.

For some numbers (m, n, k, l) no regular fault-free rectangle exists, while for other numbers there may be many. We invite the reader to try to find some of the other solutions for $(m, n, k, l) = (8, 6, 2, 2)$. The question we intend to discuss is whether there is a simple condition on m, n, k, l which guarantees the existence of a regular fault-free rectangle.

To get some insight into what conditions might be appropriate we could reverse the problem and ask: given a regular fault-free rectangle with parameters m, n, k, l, what can be deduced about m, n, k, l? To help answer this and to provide some terminology for subsequent use we shall number the m rows and n columns of the rectangle as shown in Fig. 3.

It is easy to see that $m > 2$ (because l is positive) and hence the second row is not the top row. Consequently the second row is occupied by k halves of vertical dominoes lying across its lower boundary and by k halves of vertical dominoes lying across its upper boundary. The remaining $n - 2k$

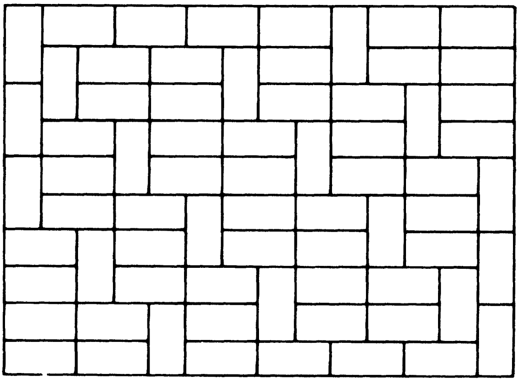

FIGURE 2.

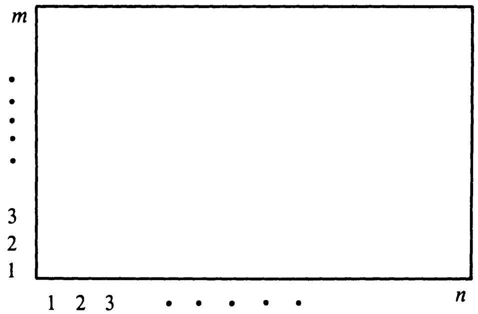

FIGURE 3.

squares must be occupied by horizontal dominoes. Thus $n - 2k$ and hence n is even. Similarly m is even. The first row is occupied by k halves of vertical dominoes lying across its upper boundary and its remaining squares must be occupied by horizontal dominoes. Thus $n - k$ is even and so k is even. Similarly l is even. An even stronger condition comes from counting, in two different ways, the crossings of fault lines by dominoes. Of the $\frac{1}{2}mn$ dominoes, $k(m - 1)$ lie across the $m - 1$ horizontal fault lines and $l(n - 1)$ lie across the $n - 1$ vertical fault lines. Hence

$$\tfrac{1}{2}mn = k(m - 1) + l(n - 1). \tag{1}$$

To summarise, for a regular fault-free rectangle with parameters m, n, k, l to exist it is necessary that

(i) m, n, k, l are all even and positive;
(ii) equation (1) should hold.

An (m, n, k, l) satisfying this condition will be called a *quad*. When we first discovered these conditions we naturally wondered if they were also sufficient, i.e. was every quad associated with an appropriate regular fault-free rectangle? Our belief in this conjecture grew stronger the more quads we considered—it always seemed that a corresponding regular fault-free rectangle could be constructed although, in our early investigations, the constructions involved a lot of trial and error. Later on, with more practice, we discovered methods which reduced the amount of backtracking required. At that point we wrote a computer program which incorporated a simple strategy for reducing the backtracking, and ran it on many quads. The program was able to find an appropriate regular fault-free rectangle in every case; but a more encouraging result was that the domino coverings which it generated exhibited many common features and a certain regularity. In particular it appeared that there was always a domino covering in which the horizontally placed dominoes occurred mostly in 'l-blocks' or '$(l - 1)$-blocks', where a 'q-block' is a set of q horizontal dominoes placed one on top of another in a $q \times 2$ rectangle. This phenomenon can be seen clearly in Fig. 2, for which $l = 4$.

Inspired by the many domino coverings generated by the computer, all of which had the above feature, we devised the following general construction applicable to any quad (m, n, k, l). In this construction the horizontal dominoes are placed first, most of them in $\frac{1}{2}n$ l-blocks and $\frac{1}{2}n - 1$ $(l - 1)$-blocks positioned alternately, the ith of these blocks (which may possibly be broken in the manner described below) being across the ith vertical fault line. To understand the precise rules for placing the blocks the reader may find it helpful to refer to Fig. 4.

The first block (an l-block) is placed in rows $1, 2, \ldots, l$. Thereafter each successive block is usually placed in the rows immediately above those occupied by the previous block. This rule fails when there are insufficient rows at the top to accommodate the next block. Suppose the ith block to be placed is a q-block ($q = l$ or $l - 1$) and there are x rows remaining ($0 \leqslant x < q$) with row numbers $m - x + 1, m - x + 2, \ldots, m$. If x is even (and note especially the case $x = 0$) we place an x-block in rows $m - x + 1$, $m - x + 2, \ldots, m$ and a $(q - x)$-block in rows $2, 3, \ldots, q - x + 1$, whereas if x is odd we place an $(x - 1)$-block in rows $m - x + 1, m - x + 2, \ldots, m - 1$ and a $(q - x + 1)$-block in rows $1, 2, \ldots, q - x + 1$; in each case these blocks are placed across the ith vertical fault line. We refer to this placing of q-dominoes as a broken q-block. The next block, across the $(i + 1)$th vertical fault line, is placed in rows $q - x + 2, q - x + 3, \ldots$. One further comment on these placings should be made: from the relation between m, n, k, l it easily follows that $m > 2l$, and this guarantees that when broken blocks cause rows $1, 2, \ldots$ to be used again there is no overlap with

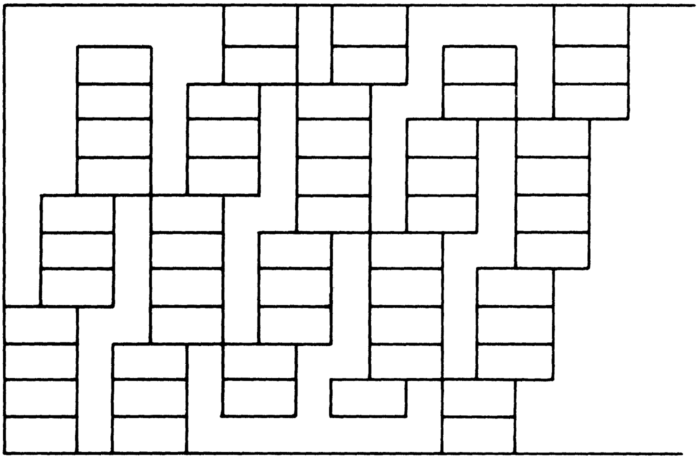

FIGURE 4. Placing the l-blocks and $(l - 1)$-blocks.

dominoes previously placed in these rows. An example of the application of these rules with $m = 12$, $l = 4$ is given in Fig. 4.

The dominoes so far placed form a number g of bands which climb upwards from left to right. In Fig. 4 the first five bands are shown. With the possible exception of the last band each one contains m, $m - 1$ or $m - 2$ dominoes. Moreover by the construction, for any two successive bands, the first has a domino in the top row if and only if the second does not have a domino in the bottom row. Consequently, by the time the right boundary of the rectangle has been reached, the total number of dominoes placed in this manner is $(g - 1)(m - 1) + h$, where h is the number of dominoes in the last band (which must end with an l-block since there is altogether an odd number of blocks). Hence

$$(g - 1)(m - 1) + h = \tfrac{1}{2}nl + (\tfrac{1}{2}n - 1)(l - 1).$$

Combining this with equation (1) and eliminating l it follows that

$$g(m - 1) - m + h = \tfrac{1}{2}(n - 2k)(m - 1).$$

Thus $m - 1$ divides $m - h$, and so $h = m$ or $h = 1$. However, $h = 1$ is impossible, for if the last band contained just one domino it would be part of a broken l-block, but broken l-blocks break into even parts. Hence $h = m$, so that the last l-block occupies rows $m - l + 1, \ldots, m$, and $g = \tfrac{1}{2}(n - 2k)$. It also follows that the number of dominoes altogether in the top and bottom rows is $g + 1$. Fig. 5 shows the construction so far for the quad (10, 14, 2, 4), except that the shaded dominoes will be placed in the next part of the construction.

So far we have ensured that the first, third, fifth, . . . vertical fault lines are crossed by precisely l dominoes, but the second, fourth, . . . vertical fault lines still require one more domino. Consider such a fault line crossed by an $(l - 1)$-block. Suppose this block has c rows lying above it and d rows lying below it. If the block is broken, one of c and d is 0 and the other is 1; but in any case, because m is even and $l - 1$ is odd, precisely one of c and d is odd. If c is odd (fault lines, 2, 4, 6 of Fig. 5) we place the extra domino in the top

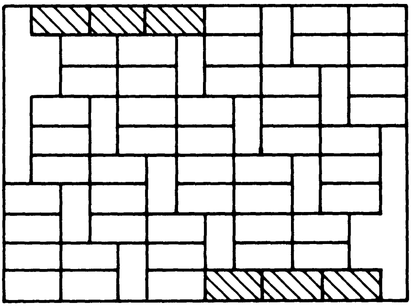

FIGURE 5.

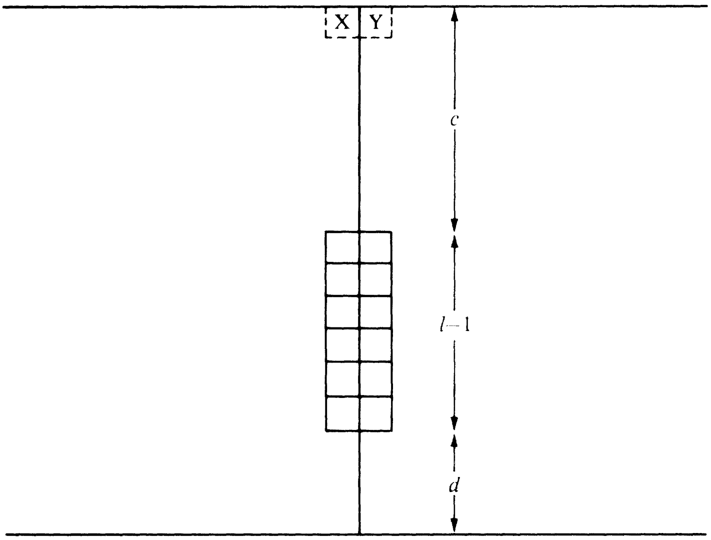

FIGURE 6.

row, whilst if d is odd we place it in the bottom row. We now show in the case c odd (the case d odd is similar) that this domino will not overlap any previously placed dominoes.

Square X (see Fig. 6) is not already covered. For if it were it would be covered by a domino belonging to an l-block crossing the previous fault line. If the l-block was unbroken we would have $d = 1$ and if it was broken one of its pieces would be a $(d - 1)$-block in rows $2, 3, \ldots, d$; both of these contradict that d is even, in the latter case because q and x (see above) are both even and $d = q - x + 1$. Furthermore, square Y is not already covered since, if it were, it would be covered by a domino in a l-block and it would follow that c was even.

Each vertical fault line is now covered by l dominoes. It is also apparent from the construction and accompanying remarks that the same pattern would have been obtained by placing the blocks in the reverse order starting from the top right-hand corner and working downwards towards the left. Hence the pattern has a rotational symmetry of order 2 (see Figs. 1, 2, 5).

The total number of dominoes in the top and bottom rows is now $g + 1 + \frac{1}{2}n - 1 = n - k$. By the rotational symmetry these are divided equally between the top and bottom rows, and it follows that in each of the top and bottom rows there are k squares still uncovered. In any other row there is one domino for each band, and consequently $n - 2g = 2k$ squares remain uncovered.

Now we shall show that in each column the unoccupied squares fall into regions of even length. This is obviously true of the first and last columns since each has a single region of $m - l$ unoccupied squares. Consider any other column. It is occupied by l halves of dominoes from an l-block, $l - 1$ halves of dominoes from an $(l - 1)$-block and a half of the extra domino used to complete the $(l - 1)$-block. If one of the two blocks is broken it is clear that the $2l$ occupied squares occur in a region at the bottom of the column and a region at the top (one of which may be empty). This leaves a single even region of $m - 2l$ unoccupied squares. If neither block is broken there will be at most two regions of unoccupied squares and, by the rule for placing the extra domino used to complete the $(l - 1)$-block, each will have even length.

In view of this the unoccupied squares can be filled by vertical dominoes, and in fact there is only one way in which they can be placed. Since the first row contains k unoccupied squares we can place k dominoes across the first horizontal fault line. This will leave $2k - k = k$ unoccupied positions in the second row. Hence k dominoes can now be placed across the second horizontal fault line, leaving k unoccupied positions in the third row. This process can be continued until each horizontal fault line is crossed by k dominoes. This completes the construction of the regular fault-free rectangle corresponding to the quad (m, n, k, l). Our investigations have therefore proved the following

THEOREM. *A regular fault-free rectangle with parameters m, n, k, l exists if and only if m, n, k, l are all positive and even and $mn = 2k(m - 1) + 2l(n - 1)$.*

We go on to establish some arithmetic properties of quads. First, $m - 1$ and $n - 1$ are coprime. For any common prime factor p would divide both sides of the basic equation (1), and so would divide (say) m; but p cannot divide both $m - 1$ and m. In particular, notice that 'squares' with $m = n$ are impossible.

Given even, positive m, n with $m - 1$ and $n - 1$ coprime, there exists at most one quad (m, n, k, l). For from equation (1) we have

$$k(m-1) \equiv \tfrac{1}{2}m \quad \operatorname{mod}(n-1),$$

and this has precisely one solution (Theorem 56 of [2]) for k in the range $0 \leqslant k \leqslant (n - 2)$. If k happens to be even and in the range $0 < k \leqslant \frac{1}{2}(n - 2)$ then l also will be even and positive, so that (m, n, k, l) is a quad. One might reasonably conclude from this that the density of pairs (m, n) for which a quad exists is, asymptotically, one-quarter of the density of odd coprime pairs—that is, by a modification of the argument giving Theorem 332 of [2], $\frac{1}{4}(2/\pi^2) = 1/(2\pi^2)$. This is indeed the case, but we omit the rather long justification.

An alternative form of the equation (1) is

$$(m - 2l)(n - 2k) = 4kl - 2k - 2l.$$

Hence given even, positive k, l there exist only finitely many quads (whose m, n correspond to the pairs of even divisors of $4kl - 2k - 2l$). There is in fact always at least one such quad, namely $(2kl - k + l, 2k + 2, k, l)$.

Finally we give in Table 1 the 36 quads with $m + n \leqslant 64$ and $m < n$, excluding on the grounds of symmetry those with $m \geqslant n$.

m	n	k	l	m	n	k	l	m	n	k	l
6	8	2	2	18	26	2	8	14	42	10	4
10	14	2	4	6	38	8	2	8	48	14	2
6	18	4	2	22	24	6	6	24	34	12	4
12	16	6	2	18	32	6	6	18	40	12	4
8	20	6	2	14	36	14	2	16	42	6	6
14	16	4	4	12	38	14	2	28	32	12	4
12	20	4	4	22	32	2	10	16	44	12	4
14	20	2	6	20	34	4	8	14	46	4	6
6	28	6	2	12	42	8	4	10	50	6	4
18	24	10	2	6	48	10	2	30	32	8	8
10	32	4	4	26	30	4	10	26	38	2	12
8	34	10	2	24	32	14	2	6	58	12	2

TABLE 1.

We are indebted to Dr J. F. Rigby for several enlightening conversations on this work.

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1. M. Gardner, Mathematical games, *Scientific American* **203**, 186–198 (November 1960).
2. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (4th edition). Oxford University Press (1960).

M. D. ATKINSON
W. F. LUNNON

Department of Computing Mathematics, University College, Cardiff

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A rise by any other name ...

"David Ennals, the Social Services Secretary, clutched at the 'single figure' hint and interpreted it as anything less than 10 per cent—9·9 recurring, to be precise." From the *Daily Mail* for 6 February 1979 (per E. Keith Lloyd).

United we stand

"The four normal channels were reduced to one—dubbed unofficially BBC-10 as the product of Radios One, Two, Three and Four." From the *Guardian* for 23 December 1978 (per Eric Primrose and D. G. H. B. Lloyd).

Sufferings of a soul in algebra

"A soul ... is easily defined negatively: it is simply what curls up and hides when there is any mention of algebraic series." From Robert Musil, *The man without qualities* (per Semantikos).